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Some properties of string field algebra

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Abstract: We examine string field algebra which is generated by star product in Witten's string field theory including ghost part. We perform calculations using oscillator representation consistently. We construct wedge like states in ghost part and investigate algebras among them. As a by-product we have obtained some solutions of vacuum string field theory. We also discuss some problems about identity state. We hope these calculations will be useful for further investigation of Witten type string field theory.

Keywords: Solitons Monopoles and Instantons, Bosonic Strings, D-branes.

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Contents

1. Introduction

Recently, string field theory has been discussed again in the context of tachyon condensation[1] and it is important to get exact solutions of it for reasonable discussion. Motivated by development of noncommutative field theory, some techniques are discussed especially to construct analogy of projectors in matter part of open string field theory with Witten type $\star$ product [2]. String field algebra with respect to the $\star$ product has been often discussed

[1]For a review of recent works, see [1] for example.

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only in matter part, but to solve equation of motion of Witten type string field theory, we should consider ghost part more seriously. It is important to develop some techniques in ghost part corresponding to those in matter part and discuss some properties for this purpose.

In this paper, we expand some techniques of string field algebra which was mainly developed in [4, 6] into ghost part by using oscillator representation consistently.[2] We can calculate the $\star$ product analogously in bc ghost nonzero modes by using some relations among Neumann coefficients similar to those in matter part, but we should treat ghost zero modes carefully. Differences of formulas between matter part and ghost part come from ghost zero modes or the balance of ghost number. As a by product of our formulations, we get some classical solutions of vacuum string field theory (VSFT)[3] in the Siegel gauge and resolve some mystery about identity state in the oscillator language.

We hope these calculations will be useful to get exact solutions of original Witten's string field theory [2] or analyze VSFT thoroughly.

This paper is organized as follows. In section 2, we review constructions of Witten's string vertex which gives the definition of $\star$ product between string fields and give some useful relations among Neumann coefficients which we use very often in later concrete calculations. In section 3, we investigate algebras among squeezed states. Especially, we define *reduced* star product which simplifies formulas and calculations of the $\star$ product in ghost part. In section 4, we get solutions of VSFT by using techniques in previous sections and discuss some issues about identity state. In section 5, we discuss some prospects and future problems. In appendix A and B, we collect our conventions and some useful formulas.

2. Witten's string vertex

Here we review constructions of Witten's 3-string vertex including ghost part, and show some relations among Neumann coefficients which we often use for later calculations of the $\star$ product in both matter and ghost part.

There are two ways to construct oscillator representation of Witten's 3-string vertex: one is purely algebraic treatment and the other is Neumann function method. Although the former has advantage to find relations among Neumann coefficients, there is a little nuisance to treat zero modes in ghost part. We relate both techniques by using a construction of [9] in which 6-string vertex in matter part leads 3-string vertex in matter and ghost part and show some relations between Neumann coefficients including ghost part.

2.1 Witten's N-string vertex in matter part

We can obtain matter N-string vertex algebraically. We follow the procedure of construction in [8]. Actually only $N = 6$ case is necessary for our purpose to define the $\star$ product.

[2]Although there is half-string formulation which was developed in [3, 4, 5] (and references therein) and used for concrete calculations of the $\star$ product in matter part, we do not use it because treatment of zero modes in both matter and ghost part is rather subtle.

[3]For a review, see [7].

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The connection condition on Witten type N-string vertex $|V_N\rangle$ is

$$
\begin{aligned}
X^{(r)}(\sigma) - X^{(r-1)}(\pi-\sigma) &= 0\,, \\
P^{(r)}(\sigma) + P^{(r-1)}(\pi-\sigma) &= 0\,, \qquad 0 \le \sigma \le \frac{\pi}{2}\,, \quad r = 1,\ldots,N\,.
\end{aligned} \tag{2.1}
$$

Using Fourier transformed coordinate:

$$
\begin{aligned}
Q_k(\sigma) &:= \frac{1}{\sqrt{N}} \sum_{r=1}^{N} \omega_N^{kr} X^{(r)}(\sigma) = Q_{k0} + \sqrt{2} \sum_{n=1}^{\infty} Q_{kn} \cos n\sigma\,, \\
P_k(\sigma) &:= \frac{1}{\sqrt{N}} \sum_{r=1}^{N} \omega_N^{kr} P^{(r)}(\sigma) = \frac{1}{\pi} \left(P_{k0} + \sqrt{2} \sum_{n=1}^{\infty} P_{kn} \cos n\sigma \right), \\
\omega_N &= \exp\left(\frac{2\pi i}{N}\right),
\end{aligned} \tag{2.2}
$$

this connection condition is rewritten as

$$
\begin{aligned}
Q_k(\sigma) &= \begin{cases} \omega_N^{k} Q_k(\pi-\sigma) & 0 \le \sigma \le \frac{\pi}{2} \\ \omega_N^{-k} Q_k(\pi-\sigma) & \frac{\pi}{2} \le \sigma \le \pi \end{cases}, \\
P_k(\sigma) &= \begin{cases} -\omega_N^{k} P_k(\pi-\sigma) & 0 \le \sigma \le \frac{\pi}{2} \\ -\omega_N^{-k} P_k(\pi-\sigma) & \frac{\pi}{2} \le \sigma \le \pi \end{cases} \quad (k = 1,\ldots,N)
\end{aligned} \tag{2.3}
$$

or

$$
(1 - Y_k)|Q_k) = (1 + Y_k)|P_k) = 0\,, \tag{2.4}
$$

where

$$
\begin{aligned}
Y_k &= \cos\left(\frac{2\pi k}{N}\right) C + \sin\left(\frac{2\pi k}{N}\right) X_{\mathrm{GJ}}\,, \quad C_{nm} = (-1)^n \delta_{n,m}\,, \qquad n,m \ge 0\,, \\
X_{\mathrm{GJ}}(\sigma,\sigma') &= i\left(\theta\left(\frac{\pi}{2} - \sigma\right) - \theta\left(\sigma - \frac{\pi}{2}\right)\right) \delta(\sigma + \sigma' - \pi)\,, \\
X_{\mathrm{GJ}} &= X_{\mathrm{GJ}}^{\dagger}\,, \quad X_{\mathrm{GJ}}^2 = 1, \bar{X}_{\mathrm{GJ}} = X_{\mathrm{GJ}}^T = -X_{\mathrm{GJ}} = C X_{\mathrm{GJ}} C\,, \\
Y_k^{\dagger} &= Y_k\,, \quad Y_k^2 = 1\,, \quad \bar{Y}_k = Y_k^T = Y_{-k} = C Y_k C\,.
\end{aligned} \tag{2.5}
$$

Under the ansatz

$$
|V_N\rangle = \mu_N e^{E_N} |0\rangle\,, \qquad E_N = -\frac{1}{2} \sum_{k=1}^{N} A_k^{\dagger} U_k A_{N-k}^{\dagger}\,, \tag{2.6}
$$

where μ_N is normalization constant and using relations of Fourier transformed creation-annihilation operators:

$$
\begin{aligned}
A_{kn} &= \frac{1}{\sqrt{N}} \sum_{r=1}^{N} \omega_N^{rk} a_n^{(r)}\,, \quad A_{kn}^{\dagger} = \frac{1}{\sqrt{N}} \sum_{r=1}^{N} \omega_N^{-rk} a_n^{(r)\dagger}\,, \quad [A_{kn}, A_{lm}^{\dagger}] = \delta_{k,l}\delta_{n,m} \quad (n,m \ge 0) \\
Q_{kn} &= \frac{i}{\sqrt{2n}} \sqrt{2\alpha'} (A_{kn} - A_{-kn}^{\dagger})\,, \ P_{kn} = \sqrt{\frac{n}{2}} \frac{1}{\sqrt{2\alpha'}} (A_{kn} + A_{-kn}^{\dagger})\,, \quad [Q_{kn}, P_{lm}] = i\delta_{k+l,0}\delta_{n,m} \\
Q_{k0} &= \frac{i}{2} \sqrt{2\alpha'} (A_{k0} - A_{-k0}^{\dagger})\,, \quad P_{kn} = \frac{1}{\sqrt{2\alpha'}} (A_{k0} + A_{-k0}^{\dagger})\,, \quad [Q_{k0}, P_{l0}] = i\delta_{k+l,0}\,,
\end{aligned} \tag{2.7}
$$

connection condition is rewritten as equations of matrices:

$$
\begin{aligned}
(1-Y_k)E(1+U_k) &= 0\,, \qquad (1+Y_k)E^{-1}(1-U_k)=0\,,\\
(1-Y_k^T)E(1+U_k^T) &= 0\,, \qquad (1+Y_k^T)E^{-1}(1-U_k^T)=0\,,
\end{aligned}
\tag{2.8}
$$

where

$$
E_{00}=\frac{1}{\sqrt{2}}\,, \qquad E_{nm}=\frac{1}{\sqrt{n}}\delta_{n,m}\,, \qquad E_{n0}=E_{0m}=0\,, \qquad n,m\geq 1\,. \tag{2.9}
$$

Its solution is given by

$$
\begin{aligned}
U_k &= (2-EY_kE^{-1}+E^{-1}Y_kE)(EY_kE^{-1}+E^{-1}Y_kE)^{-1}\\
&= (E^{-1}Y_kE+EY_kE^{-1})^{-1}(2-E^{-1}Y_kE+EY_kE^{-1})\,,\\
U_k^T &= \bar{U}_k=CU_kC=U_{N-k}\,, \qquad U_k^2=1\,, \qquad U_N=C\,.
\end{aligned}
\tag{2.10}
$$

Finally, we obtain N-string vertex as

$$
\begin{aligned}
|V_N\rangle &= \mu_N e^{E_N}|0\rangle\,,\\
E_N &= -\frac{1}{2}\sum_{r,s=1}^{N} a^{(r)\dagger}U^{rs}a^{(s)\dagger} = -\frac{1}{2}\sum_{r,s=1}^{N}\sum_{n,m\geq 0} a_n^{(r)\dagger}U_{nm}^{rs}a_m^{(s)\dagger}\,,\\
U^{rs} &= \frac{1}{N}\sum_{k=1}^{N}\omega_N^{k(r-s)}U_k\,, \qquad U_k=\sum_{t=1}^{N}\omega_N^{-kt}U^{tN}\,,\\
U^{r+1s+1} &= U^{rs}\,, \qquad U^{rsT}=U^{sr}\,, \qquad \bar{U}^{rs}=CU^{sr}C\,.
\end{aligned}
\tag{2.11}
$$

2.2 Neumann coefficients in matter part

We get relations between usual Neumann coefficients and 'Gross-Jevicki' coefficients by treating zero modes carefully.[4]

By using Neumann function method, we can construct vertex which satisfies connection condition in the following form expressed using momentum for zero modes:

$$
\begin{aligned}
|V_N\rangle &= \tilde{\mu}_N\int d^dp^{(1)}\cdots d^dp^{(N)}(2\pi)^d\delta^d(p^{(1)}+\cdots+p^{(N)})e^{\tilde{E}_N}|0,p\rangle\,,\\
\tilde{E}_N &= -\frac{1}{2}\sum_{r,s=1}^{N}\sum_{n.m\geq 1} a_n^{(r)\dagger}V_{nm}^{rs}a_m^{(s)\dagger}-\sum_{r,s=1}^{N}\sum_{n\geq 1}p^{(r)}V_{0n}^{rs}a_n^{(s)\dagger}-\frac{1}{2}\sum_{r,s=1}^{N}p^{(r)}V_{00}^{rs}p^{(s)}\,,\\
V_{nm}^{r+1,s+1} &= V_{nm}^{rs}\,, \qquad V_{nm}^{rs}=V_{mn}^{sr}\,,
\end{aligned}
\tag{2.12}
$$

where zero mode basis are given by

$$
\begin{aligned}
|0,p\rangle &= |0,p^{(1)}\rangle|0,p^{(2)}\rangle\cdots|0,p^{(N)}\rangle\,,\\
|0,p^{(r)}\rangle &= \left(\frac{2\pi}{b}\right)^{-\frac{d}{4}} e^{ip^{(r)}\hat{x}^{(r)}}e^{-\frac{1}{2}(a_0^{\dagger(r)})^2}|0\rangle\\
&= \left(\frac{2\pi}{b}\right)^{-\frac{d}{4}}\exp\left(-\frac{b}{4}p^{(r)}p^{(r)}+\sqrt{b}a_0^{(r)\dagger}p^{(r)}-\frac{1}{2}a_0^{(r)\dagger}a_0^{(r)\dagger}\right)|0\rangle\,, \qquad b=2\alpha'\,.
\end{aligned}
\tag{2.13}
$$

[4]This was done for $N=3$ case in [10, (Appendix B)].

From momentum conservation, we can redefine as $V_{0n}^{rs} \to V_{0n}^{rs} + v_n^s$. We choose V_{0n}^{rs} for simplicity such that

$$\sum_{t,s=1}^{N} (A^{-1})^{st} V_{0n}^{tr} = 0\,, \qquad A^{rs} := \frac{b}{2}\delta^{r,s} + V_{00}^{rs}\,. \tag{2.14}$$

To relate coefficients V_{nm}^{rs} with U_{nm}^{rs}, we rewrite eq. (2.12) into the form eq. (2.11) by performing gaussian integral with respect to momentum. We get the relations between U^{rs} and V^{rs}:

$$\begin{aligned}
U_{nm} &= V_{nm} - V_{n0}A^{-1}V_{0m}\,, \qquad n,m \geq 1\,,\\
U_{0n} &= \sqrt{b}A^{-1}V_{0n}\,, \qquad n \geq 1\,,\\
U_{00}^{rs} &= b\left(\frac{\sum_t \left(A^{-1}\right)^{rt}\sum_u \left(A^{-1}\right)^{su}}{\sum_{r',s'}\left(A^{-1}\right)^{r's'}} - \left(A^{-1}\right)^{rs}\right) + \delta^{r,s}\,.
\end{aligned} \tag{2.15}$$

Conversely, we get

$$\begin{aligned}
V_{00}^{rs} &= \frac{b}{2N}\sum_{k=1}^{N-1}\omega_N^{k(r-s)}\frac{1+(U_k)_{00}}{1-(U_k)_{00}} = \frac{b}{2N}\sum_{k=1}^{N-1}\cos\left(\frac{2\pi k(r-s)}{N}\right)\frac{1+(U_k)_{00}}{1-(U_k)_{00}}\,,\\
V_{0n}^{rs} &= V_{n0}^{sr} = \frac{\sqrt{b}}{N}\sum_{k=1}^{N-1}\omega_N^{k(r-s)}\frac{1}{1-(U_k)_{00}}(U_k)_{0n}\,,\\
V_{nm}^{rs} &= \frac{1}{N}C_{nm} + \frac{1}{N}\sum_{k=1}^{N-1}\omega_N^{k(r-s)}\left((U_k)_{nm} + (U_k)_{n0}\frac{1}{1-(U_k)_{00}}(U_k)_{0m}\right),
\end{aligned} \tag{2.16}$$

where the first line corresponds to the convention:[5]

$$2V_{00}^{rs} = \sum_{t=1}^{N}\sum_{n\geq 1} V_{0n}^{rt}V_{0n}^{st}\,, \qquad \sum_{t,u=1}^{N} V_{00}^{tu} = 0\,. \tag{2.17}$$

We can also rewrite V_{nm}^{rs} $(n,m \geq 1)$ like U_{nm}^{rs} $(n,m \geq 0)$ in (2.11) :

$$\begin{aligned}
V_{nm}^{rs} &= \frac{1}{N}\sum_{k=1}^{N}\omega_N^{k(r-s)}(\tilde{U}_k)_{nm}\,, \qquad n,m \geq 1\,,\\
(\tilde{U}_k)_{nm} &:= (U_k)_{nm} + (U_k)_{n0}\frac{1}{1-(U_k)_{00}}(U_k)_{0m}\,, \qquad k = 1,\ldots,N-1\,,\\
\tilde{U}_N &:= C\,,\\
\bar{\tilde{U}}_k &= \tilde{U}_k^T = C\tilde{U}_kC = \tilde{U}_{N-k}\,, \quad \tilde{U}_k^2 = 1\,.
\end{aligned} \tag{2.18}$$

In this convention for V_{0n}^{rs}, we can show

$$\sum_{t=1}^{N}\sum_{n=1}^{\infty} V_{mn}^{rt}V_{n0}^{ts} = V_{m0}^{rs}\,, \qquad \sum_{t=1}^{N}\sum_{m=1}^{\infty} V_{nm}^{rt}V_{ml}^{ts} = \delta_{n,l}\delta_{r,s}\,. \tag{2.19}$$

[5] This ambiguity comes from momentum preservation.

Finally, we get the relation between normalizations of $|V_N\rangle$:

$$\mu_N = \tilde{\mu}_N \left(\frac{(2\pi)^{N+1}}{\det_{r,s} A^{rs} \sum_{r,s=1}^{N} (A^{-1})^{rs}} \right)^{d/2} \left(\frac{2\pi}{b} \right)^{-Nd/4} . \tag{2.20}$$

2.3 From 6-string vertex to 3-string vertex including ghost part

We can construct 3-string vertex from matter 6-string vertex including ghost part.[6]

The connection condition on Witten type 3-string vertex in ghost part is

$$\begin{aligned} c^{\pm(r)}(\sigma) + c^{\pm(r-1)}(\pi - \sigma) &= 0 , \\ b^{\pm(r)}(\sigma) - b^{\pm(r-1)}(\pi - \sigma) &= 0 , \qquad 0 \le \sigma \le \frac{\pi}{2} , \qquad r = 1, 2, 3 , \end{aligned} \tag{2.21}$$

and in matter part are the same as (2.1) in $N = 3$ case. The 3-string vertex which satisfies these conditions is given as follows [9]:

$$\begin{aligned} |V_3\rangle &= \tilde{\mu}_3 \int d^d p^{(1)} d^d p^{(2)} d^d p^{(3)} (2\pi)^d \delta^d (p^{(1)} + p^{(2)} + p^{(3)}) e^{\tilde{E}_3} |0, p\rangle , \\ \tilde{E}_3 &= -\frac{1}{2} \sum_{r,s=1}^{3} \sum_{n.m \ge 1} a_n^{(r)\dagger} V_{nm}^{rs} a_m^{(s)\dagger} - \sum_{r,s=1}^{3} \sum_{n \ge 1} p^{(r)} V_{0n}^{rs} a_n^{(r)\dagger} - \frac{1}{2} \sum_{r,s=1}^{3} p^{(r)} V_{00}^{rs} p^{(s)} - \\ &\quad - \sum_{r,s=1}^{3} \sum_{n \ge 1, m \ge 0} c_{-n}^{(r)} X_{nm}^{rs} b_{-m}^{(s)} , \\ |0, p\rangle &= |0, p^{(1)}\rangle |0, p^{(2)}\rangle |0, p^{(3)}\rangle , \\ b_n^{(i)} |0, p^{(i)}\rangle &= 0 , \qquad n \ge 1 , \qquad c_m^{(i)} |0, p^{(i)}\rangle = 0 , \qquad m \ge 0 , \end{aligned} \tag{2.22}$$

where these Neumann coefficients V_{nm}^{rs}, X_{nm}^{rs} are given by using 6-string vertex in matter part as follows. In $N = 6$ case in matter part, denoting as

$$U_1 =: \tilde{U} , \qquad U_2 =: U , \qquad U_3 = -C , \qquad U_4 = \bar{U} , \qquad U_5 = \bar{\tilde{U}} , \qquad U_6 = C , \tag{2.23}$$

in the notation of section 2.1, and Neumann coefficients in section2.2 as $V^{(6)rs}_{nm}$, we get the relations :

$$\begin{aligned} V_{nm}^{rs} &= (V^{(6)rs} + V^{(6)rs+3})_{nm} = \frac{1}{3} \left(C + \omega^{r-s} U_M + \omega^{s-r} \bar{U}_M \right)_{nm} , \\ \frac{1}{\sqrt{m}} X_{mn}^{rs} \sqrt{n} &= (-1)^{r+s} (V^{(6)rs} - V^{(6)rs+3})_{mn} = \frac{1}{3} \left(-C + \omega^{s-r} U_G + \omega^{r-s} \bar{U}_G \right)_{mn} , \end{aligned} \tag{2.24}$$

where[7]

$$\begin{aligned} U_{Mmn} &= U_{mn} + U_{m0} \frac{1}{1 - U_{00}} U_{0n} , \quad U_{Gmn} = \tilde{U}_{mn} + \tilde{U}_{m0} \frac{1}{1 - \tilde{U}_{00}} \tilde{U}_{0n} , \quad m, n \ge 1 , \\ \bar{U}_M = U_M^T &= C U_M C , \quad \bar{U}_G = U_G^T = C U_G C , \quad U_M^2 = 1, \quad U_G^2 = 1 , \quad \omega = \exp \left(\frac{2\pi i}{3} \right) . \end{aligned}$$

[6] We use some results in the reference [9].

[7] We denote the matrix $(-1)^n \delta_{n,m}, \ n, m \ge 1$ as C again which leads to no confusion.

We can show

$$\sum_{t=1}^{3}\sum_{l\geq 1} V^{rt}_{nl}V^{ts}_{lm} = \delta_{nm}\delta^{rs}\,, \qquad \sum_{t=1}^{3}\sum_{l\geq 1} X^{rt}_{nl}X^{ts}_{lm} = \delta_{nm}\delta^{rs}\,, \tag{2.25}$$

which correspond to eq. (2.19). For the matrices

$$M_0 := CV^{rr},, \qquad M_\pm := CV^{rr\pm 1}\,, \qquad \tilde{M}_0 := -CX^{rr}\,, \qquad \tilde{M}_\pm := -CX^{rr\pm 1}\,, \tag{2.26}$$

where these indices run from $m = 1$ to ∞, we can show relations :

$$\begin{aligned}
&CM_0 = M_0C\,, \quad CM_+ = M_-C\,, \qquad C\tilde{M}_0 = \tilde{M}_0C\,, \quad C\tilde{M}_+ = \tilde{M}_-C\,,\\
&[M_0, M_\pm] = [M_+, M_-] = 0\,, \qquad [\tilde{M}_0, \tilde{M}_\pm] = [\tilde{M}_+, \tilde{M}_-] = 0\,,\\
&M_0 + M_+ + M_- = 1\,, \quad \tilde{M}_0 + \tilde{M}_+ + \tilde{M}_- = 1\,,\\
&M_+M_- = M_0^2 - M_0\,, \qquad \tilde{M}_+\tilde{M}_- = \tilde{M}_0^2 - \tilde{M}_0\,,\\
&M_0^2 + M_+^2 + M_-^2 = 1\,, \qquad \tilde{M}_0^2 + \tilde{M}_+^2 + \tilde{M}_-^2 = 1\,,\\
&M_0M_+ + M_+M_- + M_-M_0 = 0\,, \qquad \tilde{M}_0\tilde{M}_+ + \tilde{M}_+\tilde{M}_- + \tilde{M}_-\tilde{M}_0 = 0\,,\\
&M_\pm^2 - M_\pm = M_0M_\mp\,, \qquad \tilde{M}_\pm^2 - \tilde{M}_\pm = \tilde{M}_0\tilde{M}_\mp\,.
\end{aligned} \tag{2.27}$$

Neumann coefficients which have zero index are given by

$$\begin{aligned}
V^{rs}_{0n} &= V^{sr}_{n0} := (V^{(6)rs} + V^{(6)rs+3})_{0n} = \frac{\sqrt{b}}{3(1-U_{00})}\left(\omega^{r-s}U_{0n} + \omega^{s-r}\bar{U}_{0n}\right),\\
V^{rs}_{00} &:= (V^{(6)rs} + V^{(6)rs+3})_{00} = \frac{b}{3}\cos\left(\frac{2\pi(r-s)}{3}\right)\frac{1+U_{00}}{1-U_{00}}\,,\\
X^{rs}_{n0} &:= (-1)^{r+s}\sqrt{n}(V^{(6)rs} - V^{(6)rs+3})_{n0} = \frac{\sqrt{bn}}{3(1-\tilde{U}_{00})}\left(\omega^{r-s}\tilde{U}_{0n} + \omega^{s-r}\bar{\tilde{U}}_{0n}\right).
\end{aligned} \tag{2.28}$$

Especially, ambiguity which caused by redefinitions of $V^{(6)}{}^{rs}_{0n}$ is canceled in the formula of X^{rs}_{n0}. For these matrices, we can show some relations:[8]

$$\begin{aligned}
&\sum_{t-1}^{3}\sum_{l=1}^{\infty} V^{rt}_{nl}V^{ts}_{l0} = V^{rs}_{n0}\,, \qquad \sum_{t=1}^{3}\sum_{l=1}^{\infty} V^{rt}_{0l}V^{ts}_{l0} = 2V^{rs}_{00}\,, \qquad \sum_{t-1}^{3}\sum_{l=1}^{\infty} X^{rt}_{nl}X^{ts}_{l0} = X^{rs}_{n0}\,.\\
&CV^{r\,s}_{\ \,0} = V^{s\,r}_{\ \,0}\,, \quad \sum_{t=1}^{3} V^{t\,s}_{\ \,0} = \sum_{t=1}^{3} V^{r\,t}_{\ \,0} = 0\,, \quad CX^{r\,s}_{\ \,0} = X^{s\,r}_{\ \,0}\,, \quad \sum_{t=1}^{3} X^{t\,s}_{\ \,0} = \sum_{t=1}^{3} X^{r\,t}_{\ \,0} = 0\,,\\
&V^{21}_{\ \ 0} = \frac{3M_+ - 2}{1+3M_0}V^{11}_{\ \ 0}\,, \qquad V^{31}_{\ \ 0} = \frac{3M_- - 2}{1+3M_0}V^{11}_{\ \ 0}\,,\\
&X^{21}_{\ \ 0} = -\frac{\tilde{M}_+}{1-\tilde{M}_0}X^{11}_{\ \ 0}\,, \qquad X^{31}_{\ \ 0} = -\frac{\tilde{M}_-}{1-\tilde{M}_0}X^{11}_{\ \ 0}\,, X^{r\,s}_{\ \,0} = (\delta^{rs} + X^{rs})\frac{1}{1+X^{11}}X^{11}_{\ \ 0}\,.
\end{aligned} \tag{2.29}$$

Eqs. (2.27) and (2.29) are very useful formulas for concrete calculations of the $\star$ product in the following sections.

[8]We use the vector notation as $V^{rs}_{n0} = (V^{r\,s}_{\ \,0})_n,\ X^{rs}_{n0} = (X^{r\,s}_{\ \,0})_n$.

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For completeness we write down 3-string vertex in oscillator representation:

$$
\begin{aligned}
|V_3\rangle &= \mu_3 e^{E_3}|0\rangle\,,\\
E_3 &= -\frac{1}{2}\sum_{r,s=1}^{3}\sum_{n,m\geq 0} a_n^{(r)\dagger}U'^{rs}_{nm}a_m^{(s)\dagger} - \sum_{r,s=1}^{3}\sum_{n\geq 1,m\geq 0} c_{-n}^{(r)}X^{rs}_{nm}b_{-m}^{(s)}\,,\\
U'^{rs}_{nm} &= (U^{rs}+U^{rs+3})_{nm} = \frac{1}{3}(C+\omega^{r-s}U+\omega^{s-r}\bar{U})_{nm}\,, \qquad n,m\geq 0\,,\\
\mu_3 &= \tilde{\mu}_3\left(\frac{2^4(2\pi)^3}{3b^2}\left(\frac{1-U_{00}}{1-\frac{1}{3}U_{00}}\right)^2\right)^{d/2}\left(\frac{2\pi}{b}\right)^{-3d/4}\,,\\
|0\rangle &= |0\rangle_1|0\rangle_2|0\rangle_3\,, \qquad |0\rangle_r = |0\rangle_{M(r)}|+\rangle_{G(r)}\,,\\
|+\rangle_{G(r)} &= c_0^{(r)}c_1^{(r)}|\Omega\rangle_{G(r)}\,, \qquad a_n^{(r)}|0\rangle_{M(r)} = 0,\ n\geq 0\,,
\end{aligned}
\tag{2.30}
$$

where $|\Omega\rangle_G$ is conformal vacuum. This U'^{rs}_{nm} corresponds to (2.11) of the $N=3$ case in section 2.1.

Note that $|V_3\rangle$ has cyclic symmetry:

$$
|V_3\rangle := |1,2,3\rangle = |2,3,1\rangle = |3,1,2\rangle. \tag{2.31}
$$

2.4 Reflector and identity state

In matter part, reflector and identity state correspond to $N=2$ and $N=1$ case in section 2.1, but these are rather complicated in ghost part. Here we write down these formulas explicitly in both momentum representation and oscillator representation for matter zero mode.

2.4.1 Reflector

The reflector $|V_2\rangle$ is defined as

$$
\begin{aligned}
|V_2\rangle &= |1,2\rangle = \int d^dp^{(1)}d^dp^{(2)}\delta^d(p^{(1)}+p^{(2)})\delta(b_0^{(1)}-b_0^{(2)})e^{\tilde{E}_2}|0,p\rangle = (b_0^{(1)}-b_0^{(2)})e^{E_2}|0\rangle\,,\\
\tilde{E}_2 &= -\sum_{n,m\geq 1} a_n^{\dagger(1)}C_{nm}a_m^{\dagger(2)} + \sum_{n,m\geq 1}\left(c_{-n}^{(1)}C_{nm}b_{-m}^{(2)} + c_{-n}^{(2)}C_{nm}b_{-m}^{(1)}\right)\,,\\
E_2 &= -\sum_{n,m\geq 0} a_n^{\dagger(1)}C_{nm}a_m^{\dagger(2)} + \sum_{n,m\geq 1}\left(c_{-n}^{(1)}C_{nm}b_{-m}^{(2)} + c_{-n}^{(2)}C_{nm}b_{-m}^{(1)}\right)\,.
\end{aligned}
\tag{2.32}
$$

This is antisymmetric: $|1,2\rangle = -|2,1\rangle$. The bra state of the reflector which gives bra state from ket state generally is

$$
\begin{aligned}
\langle V_2| &= \langle 1,2| = \int d^dp^{(1)}d^dp^{(2)}\langle 0,p|e^{\tilde{E}'_2}\delta^d(p^{(1)}+p^{(2)})\delta(c_0^{(1)}+c_0^{(2)}) = \langle 0|e^{E'_2}(c_0^{(1)}+c_0^{(2)})\,,\\
\tilde{E}'_2 &= -\sum_{n,m\geq 1} a_n^{(1)}C_{nm}a_m^{(2)} - \sum_{n,m\geq 1}\left(c_n^{(1)}C_{nm}b_m^{(2)} + c_n^{(2)}C_{nm}b_m^{(1)}\right)\,,\\
E'_2 &= -\sum_{n,m\geq 0} a_n^{(1)}C_{nm}a_m^{(2)} - \sum_{n,m\geq 1}\left(c_n^{(1)}C_{nm}b_m^{(2)} + c_n^{(2)}C_{nm}b_m^{(1)}\right)\,,\\
\langle 0,p| &= {}_1\langle 0,p^{(1)}|_2\langle 0,p^{(2)}|\,.
\end{aligned}
\tag{2.33}
$$

Note that this is symmetric: $\langle 1,2| = \langle 2,1|$. Here we take bra zero mode basis as

$$
\begin{aligned}
\langle 0,p| &= \left(\frac{2\pi}{b}\right)^{-d/4} \langle 0|e^{-\frac{1}{2}a_0a_0}e^{-ip\hat{x}} = \left(\frac{2\pi}{b}\right)^{-d/4} \langle 0|e^{-\frac{1}{2}a_0a_0+\sqrt{b}pa_0-(b/4)pp}\,,\\
\langle 0| &= {}_G\langle\tilde{+}|_M\langle 0|\,, \qquad {}_G\langle\tilde{+}| = {}_G\langle\Omega|c_{-1}\,, \qquad {}_M\langle 0|a_n^\dagger = 0\,, \qquad n\geq 0\,,
\end{aligned}
\tag{2.34}
$$

where ${}_G\langle\Omega|$ is conformal vacuum. We define normalization as follows:

$$
\begin{aligned}
{}_M\langle 0|0\rangle_M &= 1\,, \qquad {}_G\langle\tilde{+}|+\rangle_G = {}_G\langle\Omega|c_{-1}c_0c_1|\Omega\rangle_G = 1\,, \qquad \langle 0|0\rangle = 1\,,\\
\langle 0,p|0,p'\rangle &= \delta^d(p-p')\,, \langle 1,2|2,3\rangle = 1_{31}\,, \qquad 1_{31}|A\rangle_1 = |A\rangle_3\,, \qquad \forall |A\rangle\,.
\end{aligned}
\tag{2.35}
$$

2.4.2 Identity state

The identity state is defined in [8] as

$$
\begin{aligned}
|I\rangle &= |V_1\rangle = \frac{1}{4i}b^+\left(\frac{\pi}{2}\right)b^-\left(\frac{\pi}{2}\right)\exp\left(\sum_{n\geq 1}(-1)^n\left(-\frac{1}{2}a_n^\dagger a_n^\dagger + c_{-n}b_{-n}\right)\right)|0\rangle\\
&= \tilde{\mu}_1(2\pi)^d e^{\tilde{E}_1}b_{-1}b_0|0,0\rangle = \mu_1 e^{E_1}b_{-1}b_0|0\rangle\,,\\
\tilde{E}_1 &= -\frac{1}{2}\sum_{n,m\geq 1}a_n^\dagger C_{nm}a_m^\dagger + \sum_{n\geq 2}(-1)^n c_{-n}b_{-n} - 2c_0\sum_{n\geq 1}(-1)^n b_{-2n} -\\
&\quad -(c_1-c_{-1})\sum_{n\geq 1}(-1)^n b_{-(2n+1)}\,,\\
E_1 &= -\frac{1}{2}\sum_{n,m\geq 0}a_n^\dagger C_{nm}a_m^\dagger + \sum_{n\geq 2}(-1)^n c_{-n}b_{-n} - 2c_0\sum_{n\geq 1}(-1)^n b_{-2n} -\\
&\quad -(c_1-c_{-1})\sum_{n\geq 1}(-1)^n b_{-(2n+1)}\,,\\
\mu_1 &= \tilde{\mu}_1(2\pi)^d\left(\frac{2\pi}{b}\right)^{-d/4}\,, \qquad b_{-1}b_0|0\rangle = |0\rangle_M|\Omega\rangle_G\,.
\end{aligned}
\tag{2.36}
$$

The corresponding bra state (or integration of string field) is[9]

$$
{}_1\langle I| := {}_{12}\langle V_2|I\rangle_2\,, \qquad \langle I| = \tilde{\mu}_1\langle 0,0|b_1 e^{\tilde{E}_1'} = \mu_1\langle 0|b_1 e^{E_1'}\,,
$$

[9]Note that this formula coincides with one given by LPP formulation [11] which is based on CFT:

$$
\begin{aligned}
{}_{\rm LPP}\langle I| &= \mu_1\,{}_M\langle 0|_G\langle\Omega|c_{-1}c_0c_1\int_{\zeta_1\zeta_0\zeta_{-1}}\exp\left(\frac{1}{2}\sum_{n,m\geq 1}\alpha_n N_{nm}\alpha_m +\right.\\
&\qquad\left. + \sum_{n\geq 2, m\geq -1}c_n\tilde{N}_{nm}b_m - \sum_{i=\pm 1,0, m\geq 1}\zeta_i M_{im}b_m\right)\,,\\
N_{nm} &= \frac{1}{nm}\oint\frac{dz}{2\pi i}z^{-n}f'(z)\oint\frac{dw}{2\pi i}w^{-m}f'(w)\frac{1}{(f(z)-f(w))^2}\,,\\
\tilde{N}_{nm} &= \oint\frac{dz}{2\pi i}z^{-n+1}(f'(z))^2\oint\frac{dw}{2\pi i}w^{-m-2}(f'(w))^{-1}\frac{-1}{f(z)-f(w)}\,,\\
M_{im} &= \oint\frac{dz}{2\pi i}z^{-m-2}(f'(z))^{-1}(f(z))^{i+1}\,,
\end{aligned}
\tag{2.37}
$$

where the map $f(z)$ is defined in [12]: $f(z) = \frac{2z}{1-z^2}$.

$$
\begin{aligned}
\tilde{E}'_1 &= -\frac{1}{2}\sum_{n,m\geq 1} a_n C_{nm} a_m - \sum_{n\geq 2}(-1)^n c_n b_n + \\
&\quad +2c_0 \sum_{n\geq 1}(-1)^n b_{2n} - (c_1 - c_{-1})\sum_{n\geq 1}(-1)^n b_{2n+1}\,, \\
E'_1 &= -\frac{1}{2}\sum_{n,m\geq 0} a_n C_{nm} a_m - \sum_{n\geq 2}(-1)^n c_n b_n + \\
&\quad +2c_0 \sum_{n\geq 1}(-1)^n b_{2n} - (c_1 - c_{-1})\sum_{n\geq 1}(-1)^n b_{2n+1}\,, \qquad \langle 0|b_1 = {}_M\langle 0|_G\langle\Omega|\,. \qquad (2.38)
\end{aligned}
$$

2.5 Witten's $\star$ product and identity state

The Witten's $\star$ product of string field is defined by using 3-string vertex (2.22) as :

$$
|A\star B\rangle_1 := {}_2\langle A|_3\langle B|1,2,3\rangle = \langle 2,4|A\rangle_4\langle 3,5|B\rangle_5|1,2,3\rangle\,. \tag{2.39}
$$

We call $\Psi_{\text{id.}}$ "*identity*" with respect to the $\star$ product if $A\star\Psi_{\text{id.}} = \Psi_{\text{id.}}\star A = A$ for any string field A. This condition can be rewritten as ${}_3\langle\Psi_{\text{id.}}|1,2,3\rangle = |1,2\rangle$ in our notation. We can check whether identity state $|I\rangle$ (2.36) is *identity* or not by straightforward calculation in our oscillator representation:

$$
\begin{aligned}
{}_3\langle I|1,2,3\rangle &= \mu_1\mu_3\left(\det(1-M_0)\right)^{-\frac{d}{2}}\det(1-\tilde{M}_0)|1,2\rangle_M|1,2\rangle'_G\,, \\
|1,2\rangle_M &= \exp\left(-\sum_{n,m\geq 0} a_n^{\dagger(1)} C_{nm} a_m^{\dagger(2)}\right)|0\rangle_{M12}\,, \\
|1,2\rangle'_G &= \left(1-2[(1-\tilde{M}_0)^{-1}X^{11}_{\;\;0}]_{\mathcal{E}}\right)\cdot \\
&\quad\cdot\left([(1-\tilde{M}_0)^{-1}X^{21}_{\;\;0}]_{\mathcal{O}}(b_0^{(1)}-b_0^{(2)}) - [(1-\tilde{M}_0)^{-1}(\tilde{M}_+ b^{\dagger(1)} + \tilde{M}_- b^{\dagger(2)})]_{\mathcal{O}}\right)\cdot \\
&\quad\cdot\exp\left(\sum_{n,m\geq 1}(c_{-n}^{(1)} C_{nm} b_{-m}^{(2)} + c_{-n}^{(2)} C_{nm} b_{-m}^{(1)})\right) e^{\Delta E}|+\rangle_{G12}\,, \\
\Delta E &= \sum_{r,s=1,2} c^{\dagger(r)}\left(X^{r3}(1-\tilde{M}_0)^{-1} C X^{3s}_{\;\;0} - X^{rs}_{\;\;0}\right) b_0^{(s)} - \\
&\quad -2\sum_{r,s=1,2} c^{\dagger(r)}\left(X^{r3}_{\;\;0} - X^{r3}(1-\tilde{M}_0)^{-1}X^{11}_{\;\;0}\right)\left(1-2[(1-\tilde{M}_0)^{-1}X^{11}_{\;\;0}]_{\mathcal{E}}\right)^{-1}\cdot \\
&\quad\cdot[(1-\tilde{M}_0)^{-1} C(X^{3s}_{\;\;0} b_0^{(s)} + X^{3s} b^{\dagger(s)})]_{\mathcal{E}} \\
&= -\left(c^{\dagger(1)} - c^{\dagger(2)}\right)\frac{1}{1-\tilde{M}_0} X^{11}_{\;\;0}\left(b_0^{(1)} - b_0^{(2)}\right)\,,
\end{aligned} \tag{2.40}
$$

where we use the notation

$$
[\;]_{\mathcal{E}} := \sum_{n=1}^{\infty}(-1)^n[\;]_{2n}\,, \qquad [\;]_{\mathcal{O}} := \sum_{n=0}^{\infty}(-1)^n[\;]_{2n+1}\,, \qquad b_n^\dagger = b_{-n}, c_n^\dagger = c_{-n}\,, \quad n\geq 1\,. \tag{2.41}
$$

We can rewrite $|1,2\rangle'_G$ as

$$
\begin{aligned}
|1,2\rangle'_G &= (1-2[(1-\tilde{M}_0)^{-1}X^{11}{}_0]_{\mathcal{E}})\cdot \\
&\cdot\Bigg(\left[\frac{1}{1-\tilde{M}_0}X^{21}{}_0-(c^{\dagger(1)}-c^{\dagger(2)})\frac{1}{1-\tilde{M}_0}X^{11}{}_0\cdot\frac{1}{1-\tilde{M}_0}(\tilde{M}_+b^{\dagger(1)}+\tilde{M}_-b^{\dagger(2)})\right]_{\mathcal{O}}\times \\
&\quad\times|1,2\rangle_G-\left[\frac{1}{1-\tilde{M}_0}(\tilde{M}_+b^{\dagger(1)}+\tilde{M}_-b^{\dagger(2)})\right]_{\mathcal{O}}|V_2^r\rangle_{12}\Bigg), \\
|V_2^r\rangle_{12} &:= e^{\sum_{n,m\geq 1}(c_{-n}^{(1)}C_{nm}b_{-m}^{(2)}+c_{-n}^{(2)}C_{nm}b_{-m}^{(1)})}|+\rangle_{G12}\,, \qquad |1,2\rangle_G=(b_0^{(1)}-b_0^{(2)})|V_2^r\rangle_{12}\,. \qquad (2.42)
\end{aligned}
$$

This shows that the identity state $|I\rangle$ is *not* identity with respect to $\star$ product although $|I\rangle$ is identity if it is restricted only in matter part.

3. String field algebra

In this section we examine string field algebra of the $\star$ product between squeezed states in both matter and ghost part by using explicit formulas in previous section.

3.1 Matter part

In this subsection, we restrict calculations in matter part. We consider only zero momentum sector for simplicity. If we include matter zero modes in oscillator representation, we can perform similar calculations because relations among Neumann coefficients are the same. Note that U^{rs}_{nm} $n,m\geq 0$ (2.11) in $N=3$ case of section 2.1 and V^{rs}_{nm} $n,m\geq 1$ (2.24) satisfy the same relations such as those in (2.27). This algebra was obtained in [4, 6] essentially, but we write down string field algebra for comparison with that in ghost part which will be discussed in the following subsection.

3.1.1 Wedge state

We define squeezed state $|n_\beta\rangle$ with parameter β which correspond to wedge state as :

$$
|n_\beta\rangle := e^{\beta a^\dagger}|n\rangle=\mu_n\exp\left(\beta a^\dagger-\frac{1}{2}a^\dagger CT_n a^\dagger\right)|0\rangle\,, \qquad (3.1)
$$

where $|n\rangle$ is given by the state which is obtained by taking $\star$ product $n-1$ times with a particular squeezed states $|2\rangle$:

$$
|n\rangle := (|2\rangle)_\star^{n-1}\,,\quad |2\rangle=\mu_2 e^{-\frac{1}{2}a^\dagger CT_2 a^\dagger}|0\rangle\,. \qquad (3.2)
$$

For simplicity, we take a matrix T_2 which satisfies

$$
CT_2=T_2C\,,\qquad T_2^T=T_2\,,\qquad [M_0,T_2]=0\,,\qquad T_2\neq 1\,. \qquad (3.3)
$$

then T_n,μ_n in eq. (3.1) are given by

$$
\begin{aligned}
T_n &= \frac{T(1-T_2T)^{n-1}+(T_2-T)^{n-1}}{(1-T_2T)^{n-1}+T(T_2-T)^{n-1}}\,, \\
\mu_n &= \mu_2\left(\mu_2\mu_3^M\det{}^{-d/2}\left(\frac{1-T}{1-T+T^2}\right)\right)^{n-2}\det{}^{d/2}\left(\frac{1-T^2}{(1-T_2T)^{n-1}+T(T_2-T)^{n-1}}\right), \\
M_0 &= \frac{T}{1-T+T^2}\,, \qquad (3.4)
\end{aligned}
$$

where μ_3^M is normalization factor of 3-string vertex in matter part and the matrix T is expressed with M_0 by solving the quadratic equation : $M_0T^2-(1+M_0)T+M_0=0$. These formulas are obtained by solving recurrence equation with respect to n [6]. Note that $|2\rangle$ is Fock vacuum $|0\rangle$ if and only if $T_2=0$.

We can show the $\star$ product formula by calculating straightforwardly :

$$|n_{\beta_1}\star m_{\beta_2}\rangle=\exp\left(-\mathcal{C}_{n_{\beta_1},m_{\beta_2}}\right)|(n+m-1)_{\beta_1\rho_{1(n,m)}+\beta_2\rho_{2(n,m)}}\rangle\,, \tag{3.5}$$

where

$$\begin{aligned}
\mathcal{C}_{n_{\beta_1},m_{\beta_2}}&=\frac{1}{2}(\beta_1,\beta_2)\frac{C}{T_{n,m}}\begin{pmatrix}M_0(1-T_m) & M_-\\ M_+ & M_0(1-T_n)\end{pmatrix}\begin{pmatrix}\beta_1^T\\ \beta_2^T\end{pmatrix}=\mathcal{C}_{m_{\beta_2C},n_{\beta_1C}}\,,\\
\rho_{1(n,m)}&=\frac{M_-+M_+T_m}{T_{n,m}}\,,\qquad \rho_{2(n,m)}=\frac{M_++M_-T_n}{T_{n,m}}\,,\qquad C\rho_{1(n,m)}=\rho_{2(m,n)}C\,,\\
T_{n,m}&=\frac{(1+T)(1-T)^2}{1-T+T^2}\times\\
&\quad\times\frac{(1-T_2T)^{n+m-2}+T(T_2-T)^{n+m-2}}{((1-T_2T)^{n-1}+T(T_2-T)^{n-1})((1-T_2T)^{m-1}+T(T_2-T)^{m-1})}\\
&=1+M_0(T_nT_m-T_n-T_m)=T_{m,n}\,.
\end{aligned} \tag{3.6}$$

We can calculate $\star$ product between states of the form $a_k^\dagger\cdots a_l^\dagger|n\rangle$ by differentiating eq. (3.5) with parameter β and setting $\beta=0$.

3.1.2 Identity and sliver state

The matter identity state is given by

$$|I\rangle_M:=\mu_I\exp\left(-\frac{1}{2}a^\dagger Ca^\dagger\right)|0\rangle\,,\qquad \mu_I\mu_3^M=\det{}^{d/2}(1-M_0)=\det{}^{d/2}\left(\frac{(1-T)^2}{1-T+T^2}\right), \tag{3.7}$$

which is "identity" with respect to the $\star$ product in matter part:

$${}_{3M}\langle I|V_3\rangle_{M123}={}_{34M}\langle V_2|I\rangle_{M4}|V_3\rangle_{M123}=|V_2\rangle_{M12}\,. \tag{3.8}$$

This identity state corresponds to $n=1$ case in eqs. (3.1) and (3.4) formally, and eqs. (3.1) and (3.5) become

$$|I_\beta\rangle=\mu_I\exp\left(\beta a^\dagger-\frac{1}{2}a^\dagger Ca^\dagger\right)|0\rangle \tag{3.9}$$

and

$$\begin{aligned}
|I_\beta\star I_{\beta'}\rangle_M&=e^{-\mathcal{C}_{1(\beta,\beta')}}|I_{\beta+\beta'}\rangle_M\,,\\
\mathcal{C}_{1(\beta,\beta')}&=\frac{1}{2}\left(\beta\frac{CM_-}{1-M_0}\beta'^T+\beta'\frac{CM_+}{1-M_0}\beta^T\right)=\beta\frac{CM_-}{1-M_0}\beta'^T=\beta'\frac{CM_+}{1-M_0}\beta^T\\
&=\mathcal{C}_{1(\beta'C,\beta C)}\,.
\end{aligned} \tag{3.10}$$

respectively.

The matter sliver state $|\Xi\rangle_M$ is given by

$$|\Xi\rangle_M := \mu_\infty \exp\left(-\frac{1}{2}a^\dagger C T a^\dagger\right)|0\rangle\,, \qquad \mu_\infty = \det{}^{d/2}\left(1-T^2\right) \tag{3.11}$$

which is normalized as

$$|\Xi\star\Xi\rangle_M = |\Xi\rangle_M\,, \qquad \mu_3^M = \det{}^{d/2}\left(\frac{1-T}{1-T+T^2}\right), \quad \mu_I = \det{}^{d/2}(1-T)\,. \tag{3.12}$$

This means $|\Xi\rangle_M$ is projection in matter part.

For consistency with $T_\infty = T$, suppose $(T-T_2)^n \to 0, (n\to\infty)$, we can fix μ_2 as

$$\mu_2 = \det{}^{d/2}(1-T_2 T)\,, \tag{3.13}$$

and we get[10]

$$\mu_n = \det{}^{d/2}\left(\frac{1-T^2}{1+T\left(\frac{T_2-T}{1-T_2T}\right)^{n-1}}\right), \qquad \mu_1=\mu_I\,, \qquad |I\rangle_M = |1\rangle,\ |\Xi\rangle_M = |\infty\rangle\,. \tag{3.14}$$

We have the $\star$ product formula corresponding to $n,m=\infty$ in eq.(3.5) :

$$\begin{aligned}
|\Xi_\beta\star\Xi_{\beta'}\rangle &= e^{-|\mathcal{C}_{\infty(\beta,\beta')}}\Xi_{\beta\rho_1+\beta'\rho_2}\rangle\,,\\
\mathcal{C}_{\infty(\beta,\beta')} &= \frac{1}{2}(\beta,\beta')\frac{C}{(1-M_0)(1+T)}\begin{pmatrix} M_0(1-T) & M_- \\ M_+ & M_0(1-T)\end{pmatrix}\begin{pmatrix}\beta^T\\ \beta'^T\end{pmatrix}\\
&= \frac{1}{2}(\beta,\beta')\frac{C}{1-T^2}\begin{pmatrix} T & \rho_1-\rho_2 T \\ \rho_2-\rho_1 T & T\end{pmatrix}\begin{pmatrix}\beta^T\\ \beta'^T\end{pmatrix} = \mathcal{C}_{\infty(\beta' C,\beta C)}\,,\\
\rho_1 &= \frac{M_- + TM_+}{(1+T)(1-M_0)}\,, \quad \rho_2 = \frac{M_+ + TM_-}{(1+T)(1-M_0)}\,,\\
\frac{M_+}{1-M_0} &= \frac{\rho_2-\rho_1 T}{1-T}\,, \qquad \frac{M_-}{1-M_0} = \frac{\rho_1-\rho_2 T}{1-T}\,.
\end{aligned} \tag{3.15}$$

Here ρ_1, ρ_2 are projection operators:

$$\rho_1^2 = \rho_1\,, \quad \rho_2^2=\rho_2\,, \quad \rho_1+\rho_2 = 1\,, \quad \rho_1\rho_2=\rho_2\rho_1 = 0\,, \quad \rho_1 C = C\rho_2\,. \tag{3.16}$$

We also write down the $\star$ product between $n=1$ and $n=\infty$ state in eq. (3.5) :

$$\begin{aligned}
|I_\beta\star\Xi_{\beta'}\rangle &= \exp\left(-\frac{1}{2}\beta C\frac{T}{1-T}\beta^T - \beta C\frac{\rho_1-\rho_2 T}{1-T}\beta'^T\right)|\Xi_{\beta\rho_1(1+T)+\beta'}\rangle\,,\\
|\Xi_\beta\star I_{\beta'}\rangle &= \exp\left(-\frac{1}{2}\beta' C\frac{T}{1-T}\beta'^T - \beta C\frac{\rho_1-\rho_2 T}{1-T}\beta'^T\right)|\Xi_{\beta+\beta'\rho_2(1+T)}\rangle\,.
\end{aligned} \tag{3.17}$$

[10] $\mu_3 = \mu_3^M$ if $T_2 = 0$ [6].

3.2 Ghost part

Here we consider string field algebra in ghost part. If we consider only ghost nonzero mode and use the Fock vacuum ${}_G\langle\tilde{+}|,\ |+\rangle_G$, we can get some similar formulas to those of matter part in previous subsection because Neumann coefficients $X^{rs}_{nm},\ n,m\geq 1$ (2.24) satisfy similar relations as (2.27). But because of ghost zero mode, the $\star$ product formulas are rather complicated than those in matter part. We introduce *reduced* product and get some useful formulas and then we consider Witten's $\star$ product formula between ghost squeezed states in the Siegel gauge.[11]

3.2.1 Reduced product

We define *reduced* product $\star^r$ by

$$|A\star^r B\rangle := {}_2\langle A^r|_3\langle B^r|V_3^r\rangle_{123}\,,\qquad \langle A^r| := \langle V_2^r|A\rangle\,, \tag{3.18}$$

where we restrict string fields $|A\rangle, |B\rangle$ such that they have no b_0, c_0 modes on the Fock vacuum $|+\rangle$. Here we introduced reduced reflector $\langle V_2^r|$ and reduced 3-string vertex $|V_3^r\rangle$ which contain no b_0, c_0 modes on the vacuum ${}_G\langle\tilde{+}|,\ |+\rangle_G$, i.e. they are related with usual reflector (2.33) and 3-string vertex (2.22) by

$${}_{12}\langle V_2| = {}_{12}\langle V_2^r|(c_0^{(1)}+c_0^{(2)})\,,\qquad |V_3\rangle_{123} = \exp\left(-\sum_{r,s=1}^{3} c^{\dagger(r)}X^{rs}_{\ \ 0}b_0^{(s)}\right)|V_3^r\rangle_{123}\,. \tag{3.19}$$

This $\star^r$ product satisfies associativity : $|(A\star^r B)\star^r C\rangle = |A\star^r(B\star^r C)\rangle$. In fact we have obtained the following formula by straightforward calculation :

$$\begin{aligned}
{}_{26}\langle V_2^r|V_3^r\rangle_{456}|V_3^r\rangle_{312} &= {}_{26}\langle V_2^r|V_3^r\rangle_{536}|V_3^r\rangle_{142} = {}_{26}\langle V_2^r|V_3^r\rangle_{146}|V_3^r\rangle_{532}\\
&= (\tilde{\mu}_3^r)^2\det(1-\tilde{M}_0^2)e^{E_{1345}}|+\rangle_{G1345}\,,\\
E_{1345} &= \sum_{s,t=1,3,4,5} c^{(s)\dagger}E^{st}b^{(t)\dagger}\,,\\
E^{11} &= E^{33}=E^{44}=E^{55}=\frac{C\tilde{M}_0}{1+\tilde{M}_0}\,,\\
E^{13} &= E^{35}=E^{41}=E^{54}=\frac{C\tilde{M}_-^2}{1-\tilde{M}_0^2}\,,\\
E^{14} &= E^{31}=E^{45}=E^{53}=\frac{C\tilde{M}_+^2}{1-\tilde{M}_0^2}\,,\\
E^{15} &= E^{34}=E^{43}=E^{51}=\frac{C\tilde{M}_+\tilde{M}_-}{1-\tilde{M}_0^2}\,.
\end{aligned} \tag{3.20}$$

We define ghost squeezed state $|n_{\xi,\eta}\rangle$ with Grassmann odd parameters ξ,η which corresponds to $|n_\beta\rangle$ (3.1) in matter part :

$$|n_{\xi,\eta}\rangle := e^{\xi b^\dagger+\eta c^\dagger}|n\rangle_G = \tilde{\mu}_n\exp\left(\xi b^\dagger+\eta c^\dagger+c^\dagger C\tilde{T}_n b^\dagger\right)|+\rangle_G\,, \tag{3.21}$$

[11] We call $|\Psi\rangle$ in the Siegel gauge if $b_0|\Psi\rangle=0$.

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where $|n\rangle_G$ is defined by the state which is obtained by taking the $\star^r$ product $n-1$ times with a particular ghost squeezed state $|2\rangle_G$:

$$|n\rangle_G = (|2\rangle_G)^{n-1}_{\star^r}\,, \qquad |2\rangle_G = \exp\left(c^\dagger C\tilde{T}_2 b^\dagger\right)|+\rangle_G\,. \tag{3.22}$$

We take a matrix $\tilde{T}_2$ which satisfies

$$C\tilde{T}_2 = \tilde{T}_2 C\,, \qquad [\tilde{M}_0, \tilde{T}_2] = 0\,, \qquad \tilde{T}_2 \neq 1\,, \tag{3.23}$$

for simplicity, and then we have obtained formulas for $\tilde{T}_n, \tilde{\mu}_n$,

$$\begin{aligned}
\tilde{T}_n &= \frac{\tilde{T}(1-\tilde{T}_2\tilde{T})^{n-1} + (\tilde{T}_2-\tilde{T})^{n-1}}{(1-\tilde{T}_2\tilde{T})^{n-1} + \tilde{T}(\tilde{T}_2-\tilde{T})^{n-1}}\,,\\
\tilde{\mu}_n &= \tilde{\mu}_2\left(\tilde{\mu}_2\tilde{\mu}_3^r \det\left(\frac{1-\tilde{T}}{1-\tilde{T}+\tilde{T}^2}\right)\right)^{n-2} \det\left(\frac{(1-\tilde{T}_2\tilde{T})^{n-1} + \tilde{T}(\tilde{T}_2-\tilde{T})^{n-1}}{1-\tilde{T}^2}\right),\\
\tilde{M}_0 &= \frac{\tilde{T}}{1-\tilde{T}+\tilde{T}^2}\,.
\end{aligned} \tag{3.24}$$

by solving the same recurrence equation as that in matter part (3.4). Here $\tilde{\mu}_3^r$ is normalization factor of reduced 3-string vertex $|V_3^r\rangle$ and $\tilde{T}$ is expressed with $\tilde{M}_0$ as a solution of the quadratic equation $\tilde{M}_0\tilde{T}^2 - (1+\tilde{M}_0)\tilde{T} + \tilde{M}_0 = 0$.

For these ghost squeezed states, we have the $\star^r$ product formula:

$$|n_{\xi,\eta} \star^r m_{\xi',\eta'}\rangle = \exp\left(-\mathcal{C}_{n_{\xi,\eta},m_{\xi',\eta'}}\right)|(n+m-1)_{\xi\tilde{\rho}_{1(n,m)}+\xi'\tilde{\rho}_{2(n,m)},\eta\tilde{\rho}^T_{1(n,m)}+\eta'\tilde{\rho}^T_{2(n,m)}}\rangle\,, \tag{3.25}$$

where

$$\begin{aligned}
\mathcal{C}_{n_{\xi,\eta},m_{\xi',\eta'}} &= (\xi,\xi')\frac{C}{\tilde{T}_{n,m}}\begin{pmatrix}\tilde{M}_0(1-\tilde{T}_m) & \tilde{M}_- \\ \tilde{M}_+ & \tilde{M}_0(1-\tilde{T}_n)\end{pmatrix}\begin{pmatrix}\eta^T \\ \eta'^T\end{pmatrix} = \mathcal{C}_{m_{\xi' C,\eta' C},n_{\xi C,\eta C}}\,,\\
\tilde{\rho}_{1(n,m)} &= \frac{\tilde{M}_- + \tilde{M}_+\tilde{T}_m}{\tilde{T}_{n,m}}\,, \qquad \tilde{\rho}_{2(n,m)} = \frac{\tilde{M}_+ + \tilde{M}_-\tilde{T}_n}{\tilde{T}_{n,m}}\,, \qquad C\tilde{\rho}_{1(n,m)} = \tilde{\rho}_{2(m,n)}C\,,\\
\tilde{T}_{n,m} &= \frac{(1+\tilde{T})(1-\tilde{T})^2}{1-\tilde{T}+\tilde{T}^2}\times\\
&\quad\times\frac{(1-\tilde{T}_2\tilde{T})^{n+m-2} + \tilde{T}(\tilde{T}_2-\tilde{T})^{n+m-2}}{((1-\tilde{T}_2\tilde{T})^{n-1} + \tilde{T}(\tilde{T}_2-\tilde{T})^{n-1})((1-\tilde{T}_2\tilde{T})^{m-1} + \tilde{T}(\tilde{T}_2-\tilde{T})^{m-1})}\\
&= 1 + \tilde{M}_0(\tilde{T}_n\tilde{T}_m - \tilde{T}_n - \tilde{T}_m) = \tilde{T}_{m,n}\,.
\end{aligned} \tag{3.26}$$

These formulas are similar to those in matter part (3.5) and (3.6). We can obtain $\star^r$ product between the states of the form $b_k^\dagger \cdots c_l^\dagger|n\rangle_G$ by differentiating eq. (3.25) with respect to parameters ξ,η and setting them zero.

3.2.2 Identity and sliver like state

We consider a particular state $|I^r\rangle$ which was excluded as a candidate for $|2\rangle_G$:

$$|I^r\rangle_G := \tilde{\mu}_I^r \exp\left(c^\dagger C b^\dagger\right)|+\rangle_G\,, \qquad \tilde{\mu}_I^r\tilde{\mu}_3^r = \det{}^{-1}\left(1-\tilde{M}_0\right) = \det\left(\frac{1-\tilde{T}+\tilde{T}^2}{(1-\tilde{T})^2}\right). \tag{3.27}$$

This is identity-like state of ghost part,i.e., identity with respect to the $\star^r$ product which was introduced in eq. (3.18) because it satisfies the following equation.[12]

$$ {}_3\langle I^r|V_3^r\rangle_{123} = {}_{34}\langle V_2^r|I^r\rangle_4|V_3^r\rangle_{123} = |V_2^r\rangle_{12}\,, \tag{3.28} $$

although it is *different* from the identity state $|I\rangle$ (2.36) in ghost part as :

$$ |I\rangle = \frac{1}{4i}b^+\left(\frac{\pi}{2}\right)b^-\left(\frac{\pi}{2}\right)|I\rangle_M|I^r\rangle_G = [b^\dagger]_{\mathcal{O}}(b_0+2[b^\dagger]_{|\mathcal{E}})|I\rangle_M|I^r\rangle_G\,, \tag{3.29} $$

where we used the notation of (2.41). Corresponding to (3.9) in matter part, we consider the state of the form

$$ |I_{\xi,\eta}\rangle = \tilde{\mu}_I^r \exp\left(\xi b^\dagger + \eta c^\dagger + c^\dagger C b^\dagger\right)|+\rangle_G\,, \tag{3.30} $$

then we have the $\star^r$ product formula between them :

$$ \begin{aligned} |I_{\xi,\eta}\star^r I_{\xi',\eta'}\rangle &= e^{-\mathcal{C}_{1(\xi,\eta;\xi',\eta')}}|I_{\xi+\xi',\eta+\eta'}\rangle\,, \\ \mathcal{C}_{1(\xi,\eta;\xi',\eta')} &= \xi\frac{C\tilde{M}_-}{1-\tilde{M}_0}\eta'^T + \xi'\frac{C\tilde{M}_+}{1-\tilde{M}_0}\eta^T = \mathcal{C}_{1(\xi'C,\eta'C;\xi C,\eta C)}\,. \end{aligned} \tag{3.31} $$

This is $n=m=1$ case of eq. (3.25) formally.

Next we define the sliver-like state in ghost part :

$$ |\Xi^r\rangle_G := \tilde{\mu}_\infty \exp\left(c^\dagger C\tilde{T}b^\dagger\right)|+\rangle_G\,, \qquad \tilde{\mu}_\infty = \det{}^{-1}\left(1-\tilde{T}^2\right). \tag{3.32} $$

This state is analogy of projection with respect to the $\star^r$ product :

$$ |\Xi^r\star^r\Xi^r\rangle_G = |\Xi^r\rangle_G\,, \qquad \tilde{\mu}_3^r = \det{}^{-1}\left(\frac{1-\tilde{T}}{1-\tilde{T}+\tilde{T}^2}\right), \qquad \tilde{\mu}_I^r = \det{}^{-1}(1-\tilde{T})\,. \tag{3.33} $$

For formal consistency with the case $n=\infty$ in eqs. (3.21) (3.24), suppose $(\tilde{T}-\tilde{T}_2)^n \to 0$ $(n\to\infty)$, we fix $\tilde{\mu}_2$ as

$$ \tilde{\mu}_2 = \det{}^{-1}(1-\tilde{T}_2\tilde{T}), \tag{3.34} $$

and then we get[13]

$$ \tilde{\mu}_n = \det{}^{-1}\left(\frac{1-\tilde{T}^2}{1+\tilde{T}\left((\tilde{T}_2-\tilde{T})/(1-\tilde{T}_2\tilde{T})\right)^{n-1}}\right), \quad \tilde{\mu}_1 = \tilde{\mu}_I^r\,, \quad |I^r\rangle_G = |1\rangle_G,\ |\Xi^r\rangle = |\infty\rangle_G\,. \tag{3.35} $$

We have obtained the $\star^r$ product formula for $n,m=\infty$ case in eq. (3.25) :

$$ |\Xi_{\xi,\eta}\star^r\Xi_{\xi',\eta'}\rangle = e^{-\mathcal{C}_{\infty(\xi,\eta;\xi',\eta')}}|\Xi_{\xi\tilde{\rho}_1+\xi'\tilde{\rho}_2,\eta\tilde{\rho}_1^T+\eta'\tilde{\rho}_2^T}\rangle\,, $$

---

[12] Here the ket $|V_2^r\rangle$ is reduced reflector which was defined in eq. (2.42). Note that ${}_{12}\langle V_2^r|V_2^r\rangle_{23} = 1_{31}$ if we consider string fields which have no b_0, c_0 modes on the vacuum ${}_G\langle\tilde{+}|,\ |+\rangle_G$.

[13] $\tilde{\mu}_3 = \tilde{\mu}_3^r$ if $\tilde{T}_2 = 0$.

$$
\begin{aligned}
\mathcal{C}_{\infty(\xi,\eta;\xi',\eta')} &= (\xi,\xi')\frac{C}{(1-\tilde{M}_0)(1+\tilde{T})}\begin{pmatrix}\tilde{M}_0(1-\tilde{T}) & \tilde{M}_- \\ \tilde{M}_+ & \tilde{M}_0(1-\tilde{T})\end{pmatrix}\begin{pmatrix}\eta^T \\ \eta'^T\end{pmatrix} \\
&= (\xi,\xi')\frac{C}{1-\tilde{T}^2}\begin{pmatrix}\tilde{T} & \tilde{\rho}_1-\tilde{\rho}_2\tilde{T} \\ \tilde{\rho}_2-\tilde{\rho}_1\tilde{T} & \tilde{T}\end{pmatrix}\begin{pmatrix}\eta^T \\ \eta'^T\end{pmatrix} = \mathcal{C}_{\infty(\xi'C,\eta'C;\xi C,\eta C)}\,, \\
\tilde{\rho}_1 &= \frac{\tilde{M}_- + \tilde{T}\tilde{M}_+}{(1+\tilde{T})(1-\tilde{M}_0)},\ \tilde{\rho}_2 = \frac{\tilde{M}_+ + \tilde{T}\tilde{M}_-}{(1+\tilde{T})(1-\tilde{M}_0)}\,, \\
\frac{\tilde{M}_+}{1-\tilde{M}_0} &= \frac{\tilde{\rho}_2-\tilde{\rho}_1\tilde{T}}{1-\tilde{T}},\ \frac{\tilde{M}_-}{1-\tilde{M}_0} = \frac{\tilde{\rho}_1-\tilde{\rho}_2\tilde{T}}{1-\tilde{T}}\,.
\end{aligned} \tag{3.36}
$$

In this notation, $\tilde{\rho}_1, \tilde{\rho}_2$ are projection operators :

$$
\tilde{\rho}_1^2 = \tilde{\rho}_1\,,\quad \tilde{\rho}_2^2 = \tilde{\rho}_2,\ \tilde{\rho}_1+\tilde{\rho}_2 = 1\,,\quad \tilde{\rho}_1\tilde{\rho}_2 = \tilde{\rho}_2\tilde{\rho}_1 = 0\,,\quad C\tilde{\rho}_1 = \tilde{\rho}_2 C\,. \tag{3.37}
$$

We also write down the $\star^r$ product between the identity and sliver like states :

$$
\begin{aligned}
&|I_{\xi,\eta}\star^r \Xi_{\xi',\eta'}\rangle = \\
&\quad = \exp\left(-\xi C\frac{\tilde{T}}{1-\tilde{T}}\eta^T - \xi C\frac{\tilde{\rho}_1-\tilde{\rho}_2\tilde{T}}{1-\tilde{T}}\eta'^T - \xi' C\frac{\tilde{\rho}_2-\tilde{\rho}_1\tilde{T}}{1-\tilde{T}}\eta^T\right)\times \\
&\quad\quad \times|\Xi_{\xi\tilde{\rho}_1(1+\tilde{T})+\xi',\eta(\tilde{\rho}_1(1+\tilde{T}))^T+\eta'}\rangle\,, \\
&|\Xi_{\xi,\eta}\star^r I_{\xi',\eta'}\rangle = \\
&\quad = \exp\left(-\xi' C\frac{\tilde{T}}{1-\tilde{T}}\eta'^T - \xi C\frac{\tilde{\rho}_1-\tilde{\rho}_2\tilde{T}}{1-\tilde{T}}\eta'^T - \xi' C\frac{\tilde{\rho}_2-\tilde{\rho}_1\tilde{T}}{1-\tilde{T}}\eta^T\right)\times \\
&\quad\quad \times|\Xi_{\xi+\xi'\tilde{\rho}_2(1+\tilde{T}),\eta+\eta'(\tilde{\rho}_2(1+\tilde{T}))^T}\rangle\,.
\end{aligned} \tag{3.38}
$$

3.2.3 The $\star$ product in ghost part

Here we consider Witten's $\star$ product defined by eq. (2.39) in the Siegel gauge. By using the reduced product $\star^r$ (3.18), the $\star$ product (2.39) in the Siegel gauge can be written rather simply, i.e. for $|\Phi\rangle = b_0|\phi\rangle, |\Psi\rangle = b_0|\psi\rangle$, where $|\phi\rangle, |\psi\rangle$ have no b_0, c_0 modes, their $\star$ product is[14]

$$
\begin{aligned}
|\Phi\star\Psi\rangle &= |\phi\star^r\psi\rangle + b_0\left({}_2\langle\phi^r|_3\langle\psi^r|\sum_{s=1}^{3}c^{(s)\dagger}X^{s1}_{\ 0}|V_3^r\rangle_{123}\right) \\
&= (1+b_0c^\dagger X^{11}_{\ 0})|\phi\star^r\psi\rangle + b_0\sum_{s=2,3}{}_2\langle\phi^r|_3\langle\psi^r|c^{(s)\dagger}X^{s1}_{\ 0}|V_3^r\rangle_{123}.
\end{aligned} \tag{3.39}
$$

From this formula (3.39) and eq. (3.25), we have obtained the $\star$ product between ghost squeezed states (3.21) in the Siegel gauge :

$$
\begin{aligned}
&|(b_0 n_{\xi,\eta})\star(b_0 m_{\xi',\eta'})\rangle = \\
&= \left(1+b_0\left(c^\dagger X^{11}_{\ 0} + \left(\xi C + \frac{\partial}{\partial\eta}\tilde{T}_n\right)X^{21}_{\ 0} + \left(\xi' C + \frac{\partial}{\partial\eta'}\tilde{T}_m\right)X^{31}_{\ 0}\right)\right)|n_{\xi,\eta}\star^r m_{\xi',\eta'}\rangle \\
&= \left(1+b_0c^\dagger\frac{1-\tilde{T}_n\tilde{T}_m}{\tilde{T}_{n,m}}X^{11}_{\ 0} - b_0(\xi\tilde{\rho}_{1(n,m)}+\xi'\tilde{\rho}_{2(n,m)})\frac{1}{1-\tilde{M}_0}X^{11}_{\ 0}\right)|n_{\xi,\eta}\star^r m_{\xi',\eta'}\rangle\,,
\end{aligned} \tag{3.40}
$$

[14] We supposed that $|\phi\rangle, |\psi\rangle$ are Grassmann even as $|n\rangle_G$ in this formula.

where

$$\frac{1-\tilde{T}_n\tilde{T}_m}{\tilde{T}_{n,m}} = \frac{1-\tilde{T}}{1-\tilde{M}_0}\cdot\frac{(1-\tilde{T}_2\tilde{T})^{n+m-2}-(\tilde{T}_2-\tilde{T})^{n+m-2}}{(1-\tilde{T}_2\tilde{T})^{n+m-2}+\tilde{T}(\tilde{T}_2-\tilde{T})^{n+m-2}}\,. \tag{3.41}$$

In particular, for the identity and sliver like states we get some rather simple formulas:

$$\begin{aligned}
|b_0 I^r \star b_0 I^r\rangle &= |I^r\rangle\,,\\
|b_0 \Xi^r \star b_0 \Xi^r\rangle &= |b_0 I^r \star b_0 \Xi^r\rangle = |b_0 \Xi^r \star b_0 I^r\rangle = \left(1+b_0 c^\dagger \frac{1-\tilde{T}}{1-\tilde{M}_0}X^{11}{}_0\right)|\Xi^r\rangle\,.
\end{aligned} \tag{3.42}$$

and with parameters ξ,η,

$$\begin{aligned}
&|b_0 I_{\xi,\eta}\star b_0 I_{\xi',\eta'}\rangle = \left(1-b_0(\xi+\xi')\frac{1}{1-\tilde{M}_0}X^{11}{}_0\right)e^{-\mathcal{C}_1(\xi,\eta;\xi',\eta')}|I_{\xi+\xi',\eta+\eta'}\rangle\,,\\
&|b_0\Xi_{\xi,\eta}\star b_0\Xi_{\xi',\eta'}\rangle =\\
&\quad = \left(1+b_0 c^\dagger\frac{1-\tilde{T}}{1-\tilde{M}_0}X^{11}{}_0 - b_0(\xi\tilde{\rho}_1+\xi'\tilde{\rho}_2)\frac{1}{1-\tilde{M}_0}X^{11}{}_0\right)e^{-\mathcal{C}_\infty(\xi,\eta;\xi',\eta)}|\Xi_{\xi\tilde{\rho}_1+\xi'\tilde{\rho}_2,\eta\tilde{\rho}_1^T+\eta'\tilde{\rho}_2^T}\rangle\,,\\
&|b_0 I_{\xi,\eta}\star b_0\Xi_{\xi',\eta'}\rangle = \left(1+b_0 c^\dagger\frac{1-\tilde{T}}{1-\tilde{M}_0}X^{11}{}_0 - b_0(\xi\tilde{\rho}_1(1+\tilde{T})+\xi')\frac{1}{1-\tilde{M}_0}X^{11}{}_0\right)|I_{\xi,\eta}\star^r\Xi_{\xi',\eta'}\rangle,\\
&|b_0\Xi_{\xi,\eta}\star b_0 I_{\xi',\eta'}\rangle = \left(1+b_0 c^\dagger\frac{1-\tilde{T}}{1-\tilde{M}_0}X^{11}{}_0 - b_0(\xi+\xi'\tilde{\rho}_2(1+\tilde{T}))\frac{1}{1-\tilde{M}_0}X^{11}{}_0\right)|\Xi_{\xi,\eta}\star^r I_{\xi',\eta'}\rangle\,.
\end{aligned} \tag{3.43}$$

4. Solutions of vacuum string field theory

We consider equation of motion of VSFT [7] for an application of techniques which we have developed in previous sections.

The equation of motion of VSFT is given by

$$Q|\Psi\rangle + |\Psi\star\Psi\rangle = 0\,, \tag{4.1}$$

where Q is given by a linear combination of c ghost modes :

$$Q = c_0 + \sum_{n=1}^{\infty} f_n\left(c_n + (-1)^n c_n^\dagger\right) = c_0 + f\cdot\left(c + C c^\dagger\right). \tag{4.2}$$

Here f_ns are particular coefficients which we choose appropriately. To solve eq. (4.1), we set the ansatz for solutions in the Siegel gauge as :

$$|\Psi\rangle = b_0|P\rangle_M\left(\sum_{n=1}^{\infty} g_n|n\rangle_G\right),\qquad |P\star P\rangle_M = |P\rangle_M\,. \tag{4.3}$$

This means that it is factorized into matter and ghost part, matter part is some projector and ghost part is some linear combination of ghost squeezed states of the form (3.21) with appropriate coefficients g_n for simplicity. In matter part, we can choose $|P\rangle_M$ as matter identity $|I\rangle_M$ or matter sliver state $|\Xi\rangle_M$ for example.[15] To solve eq. (4.1), we substitute (4.3) and find coefficients g_n of $|\Psi\rangle$ and f_n of Q in the same time.

Noting

$$Q|n\rangle_G = c^\dagger C(1-\tilde{T}_n)\cdot f^T|n\rangle_G\,, \tag{4.4}$$

we have obtained the following solutions.

1. identity-like solution

$$Q = c_0\,,\qquad |\Psi\rangle = -b_0|P\rangle_M|I^r\rangle_G\,. \tag{4.5}$$

2. sliver-like solution

$$Q = c_0-(c+c^\dagger)\frac{1}{1-\tilde{M}_0}X^{11}_{\ \ 0}\,,\qquad |\Psi\rangle = -b_0|P\rangle_M|\Xi^r\rangle_G\,. \tag{4.6}$$

   This was constructed in [13].[16]

3. another solution

$$Q = c_0-(c+c^\dagger)\frac{1}{1-\tilde{M}_0}X^{11}_{\ \ 0}\,,\qquad |\Psi\rangle = -b_0|P\rangle_M(|I^r\rangle_G-|\Xi^r\rangle_G)\,. \tag{4.7}$$

   This is a solution for the Q which is the same as the above one.

The chosen Qs for these solutions consist of even modes of c ghost because

$$(X^{11}_{\ \ 0})_{2n+1} = 0\,,\qquad (\tilde{M}_0)_{2n,2m+1} = (\tilde{M}_0)_{2m+1,2n} = 0 \tag{4.8}$$

which are obtained from eqs. (2.27) and (2.29). Although these Q do not vanish on the identity state $\langle I|$ (2.38) :

$$\langle I|c_0 = \langle I|\frac{(-1)^n}{2}(c_{2n}+c^\dagger_{2n})\neq 0 \tag{4.9}$$

we have obtained

$${}_3\langle I|c_0^{(3)}|V_3\rangle_{123} = 0 \tag{4.10}$$

by straightforward calculation. This means that we could use Q which consists of an arbitrary linear combination of $c_0, (c_{2n}+c^\dagger_{2n})$ for the definition of VSFT in our oscillator formulation although it was afraid to be used if $\langle I|Q\neq 0$ in [7].

Note that the equation (4.10) also means

$$|(c_0I)\star A\rangle = 0\,,\qquad \forall|A\rangle\,. \tag{4.11}$$

If we take identity state $|I\rangle$ (2.36) as $|A\rangle$, this becomes $|(c_0I)\star I\rangle = 0$. If identity state $|I\rangle$ were identity with respect to the $\star$ product, this might show $|c_0I\rangle = 0$, but it is inconsistent with the fact $c_0|I\rangle\neq 0$. We can solve this mystery which was referred in [12] in our oscillator formulation. Because identity state $|I\rangle$ is not identity as was shown in section 2.5, we have $|(c_0I)\star I\rangle\neq c_0|I\rangle$ consistently. Eq. (4.11) also follows from the fact

$$(c_0^{(1)}+c_0^{(2)}+c_0^{(3)})|V_3\rangle = 0\,,\qquad c_0|I\star A\rangle = |I\star(c_0A)\rangle\,,\qquad \forall A \tag{4.12}$$

which we can check from eqs. (2.22) and (2.40).

[15] If we use half-string formulation [3, 4, 5], we can choose other projector in matter part.

[16] This formula is simpler than that in [13].

5. Discussion

In this paper we have constructed some squeezed states $|n_{\xi,\eta}\rangle$ with parameters ξ, η which correspond to wedge states in matter part. They satisfy rather simple algebra with respect to the reduced $\star^r$ product which we introduced for convenience. We have also obtained the $\star$ product formula between them in the Siegel gauge.

For ghost squeezed states, the $\star^r$ product formulas are very similar to those in matter part, but the $\star$ product is a little complicated by ghost zero mode. In these calculations, the algebras which Neumann coefficients satisfy are essential. We have shown them first with some review for self-contained.

By applying our formulations to VSFT, we have obtained some solutions of equation of motion which are expressed by the ghost identity and sliver like states. To get some physical implications, we should examine these solutions as was done for a particular solution in [13]. It is a future problem to discuss physical spectrum around these solutions, potential hight and so on. We hope that our techniques will be helpful to get solutions of original Witten's string field theory and analyze them.

Although we have used oscillator representation consistently and performed algebraic calculations only, it is also important to interpret them in the CFT language to obtain geometrical meaning of our treatment in ghost part.

We noted that the identity state $|I\rangle$ is not identity with respect to the $\star$ product by straightforward calculation. We can also show that associativity of the $\star$ product is broken generally by calculating ${}_{26}\langle V_2|V_3\rangle_{456}|V_3\rangle_{312}$ straightforwardly although the reduced $\star^r$ product satisfies associativity. This comes from Neumann coefficients ${X^{r}}^{s}_{0}$ which are related to ghost zero mode. This problem which is inconsistent with naive consideration about connection condition was known as associativity anomaly [14]. It is subtle but important problem to discuss Witten type string field theory rigorously and consistently.[17]It is a future problem to investigate this issue more seriously.

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A. Conventions

Here we list up our conventions.

[17]It was not necessary to consider associativity of the $\star$ product in this paper because we discussed only equation of motion.

Mode expansion in matter part ($\mu = 1, \cdots, d$):

$$X^\mu(\sigma) \;=\; x_0^\mu + \sqrt{2}\sum_{n=1}^{\infty} x_n^\mu \cos n\sigma = x_0^\mu + i\sqrt{2\alpha'}\sum_{n=1}^{\infty}\frac{1}{n}(\alpha_n^\mu - \alpha_{-n}^\mu)\cos n\sigma\,, \tag{A.1}$$

$$P_\mu(\sigma) \;=\; \frac{1}{\pi}\left(p_{0\mu} + \sum_{n=1}^{\infty} p_{n\mu}\cos n\sigma\right) = \frac{1}{\pi}\left(p_{0\mu} + \frac{1}{\sqrt{2\alpha'}}\sum_{n=1}^{\infty}\eta_{\mu\nu}(\alpha_n^\nu + \alpha_{-n}^\nu)\cos n\sigma\right).$$

Matter nonzero mode ($n \geq 1$):

$$\alpha_n^\mu \;=\; \sqrt{n}a_n^\mu\,, \qquad \alpha_{-n}^\mu = \sqrt{n}a_n^{\dagger\mu}\,, \qquad [\alpha_n^\mu, \alpha_m^\nu] = n\delta_{n+m,0}\eta^{\mu\nu}\,, \qquad [a_n^\mu, a_m^{\dagger\nu}] = \delta_{n,m}\eta^{\mu\nu}\,,$$

$$x_n^\mu \;=\; \frac{i}{\sqrt{2n}}\sqrt{2\alpha'}(a_n^\mu - a_n^{\dagger\mu})\,, \qquad p_{n\mu} = \sqrt{\frac{n}{2}}\frac{\eta_{\mu\nu}}{\sqrt{2\alpha'}}(a_n^\nu + a_n^{\dagger\nu})\,, \qquad [x_n^\mu, p_{m\nu}] = i\delta_{n,m}\delta_\nu^\mu\,.$$

Matter zero mode:

$$x_0^\mu = \frac{i}{2}\sqrt{2\alpha'}(a_0^\mu - a_0^{\dagger\mu})\,, \quad p_{0\mu} = \frac{1}{\sqrt{2\alpha'}}\eta_{\mu\nu}(a_0^\nu + a_0^{\dagger\nu})\,, \quad [x_0^\mu, p_{0\nu}] = i\delta_\nu^\mu\,, \quad [a_0^\mu, a_0^{\dagger\nu}] = \eta^{\mu\nu}\,. \tag{A.2}$$

Mode expansion of bc ghost :

$$c^\pm(\sigma) = \sum_{n=-\infty}^{\infty} c_n e^{\mp in\sigma}\,, \qquad b^\pm(\sigma) = \sum_{n=-\infty}^{\infty} b_n e^{\mp in\sigma}\,, \quad \{c_n, b_m\} = \delta_{n+m,0}\,. \tag{A.3}$$

B. Some formulas

Here we collect some useful formulas which we often use in calculations.

B.1 Gaussian integral

We can prove useful formulas for oscillators by using coherent state and performing Gaussian integral.

For bosonic oscillator $a_n, a_n^\dagger$ and Fock vacuum $|0\rangle$:

$$[a_m, a_n^\dagger] = \delta_{mn}\,, \qquad a_n|0\rangle = 0\,, \tag{B.1}$$

we get a formula

$$\begin{aligned}
&\exp\left(\frac{1}{2}aMa + \lambda a\right)\exp\left(\frac{1}{2}a^\dagger N a^\dagger + \mu a^\dagger\right)|0\rangle = \\
&= \frac{1}{\sqrt{\det(1-MN)}}\exp\left(\frac{1}{2}\lambda N(1-MN)^{-1}\lambda + \frac{1}{2}\mu M(1-NM)^{-1}\mu + \lambda(1-NM)^{-1}\mu\right)\cdot \\
&\quad\cdot\exp\left((\lambda N + \mu)(1-MN)^{-1}a^\dagger + \frac{1}{2}a^\dagger N(1-MN)^{-1}a^\dagger\right)|0\rangle\,,
\end{aligned} \tag{B.2}$$

where M, N are symmetric matrices.

For fermionic oscillator[18] $b_n, c_n, b_n^\dagger, c_n^\dagger$ and Fock vacuum $|+\rangle_G$:

$$\{b_m, c_n^\dagger\} = \delta_{mn}\,, \quad \{c_m, b_n^\dagger\} = \delta_{mn}\,, \quad c_n|+\rangle_G = b_n|+\rangle_G = 0\,, \tag{B.3}$$

[18] We often use the notation : $b_{-n} = b_n^\dagger, c_{-n} = c_n^\dagger,\ n \geq 1$.

we get[19]

$$
\begin{aligned}
&\exp\left(bMc + b\lambda + \mu c\right) \exp\left(b^\dagger N c^\dagger + b^\dagger \xi + \eta c^\dagger\right) |+\rangle_G = \\
&= \det(1+MN) \exp\left((\mu - \eta M)(1+NM)^{-1}\xi - (\mu N + \eta)(1+MN)^{-1}\lambda\right) \cdot \\
&\quad \cdot \exp\left(b^\dagger (1+NM)^{-1}(-N\lambda + \xi) + (\mu N + \eta)(1+MN)^{-1} c^\dagger + b^\dagger N (1+MN)^{-1} c^\dagger\right) \times \\
&\quad \times |+\rangle_G \,,
\end{aligned}
\tag{B.4}
$$

where M, N are Grassmann even and λ, μ, ξ, η are odd.

B.2 Coherent states for bc-ghost

In our concrete calculations which contain both bc ghost nonzero and zero modes, it is useful to insert completeness formula for coherent states.

Coherent states of nonzero mode:

$$
\begin{aligned}
|\xi\rangle &:= e^{-\xi^c b^\dagger - \xi^b c^\dagger} |+\rangle_G \,, \qquad \langle \xi| := {}_G\langle \tilde{+}| e^{-b\bar{\xi}^c - c\bar{\xi}^b} \,, \\
b_n|\xi\rangle &= \xi^b_n |\xi\rangle \,, \qquad \langle \xi| b^\dagger_n = \langle \xi| \bar{\xi}^b_n \,, \qquad c_n|\xi\rangle = \xi^c_n|\xi\rangle \,, \qquad \langle \xi| c^\dagger_n = \langle \xi|\bar{\xi}^c_n \,.
\end{aligned}
\tag{B.5}
$$

Coherent state of zero mode:

$$
\begin{aligned}
|\xi_0\rangle &:= e^{-\xi_0 b_0} |+\rangle_G \,, \qquad \langle \xi_0| := {}_G\langle \tilde{+}| e^{-c_0 \bar{\xi}_0} \,, \\
c_0 |\xi_0\rangle &= \xi_0 |\xi_0\rangle \,, \qquad \langle \xi_0| b_0 = \langle \xi_0| \bar{\xi}_0 \,.
\end{aligned}
\tag{B.6}
$$

Completeness formula:

$$
1 = \int d\bar{\xi} d\xi d\bar{\xi}_0 d\xi_0 |\xi, \xi_0\rangle e^{-\bar{\xi}^c \xi^b - \bar{\xi}^b \xi^c - \bar{\xi}_0 \xi_0} \langle \xi, \xi_0| \,. \tag{B.7}
$$

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[19] Note that this formula contains no b_0, c_0.

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